[11:00-11:30] Beat notes and periodic signals

When two sinusoidal signals $x_1(t) = A_1 \cos(\omega_1 t + \phi_1)$ and $x_2(t) = A_2 \cos(\omega_2 t + \phi_2)$ are multiplied together, the result contains the sum and difference frequencies:

$$x_3(t) = A_{\text{sum}} \cos\big((\omega_1 + \omega_2)t + \phi_{\text{sum}}\big) + A_{\text{diff}} \cos\big((\omega_2 - \omega_1)t + \phi_{\text{diff}}\big)$$

Example:

$$\begin{aligned}
x(t) &= \cos(2\pi 10t)\sin(2\pi 1000t) \\
&= \left(\frac{e^{j2\pi 10t} + e^{-j2\pi 10t}}{2}\right)\left(\frac{e^{j2\pi 1000t} - e^{-j2\pi 1000t}}{2j}\right) \\
&= \frac{1}{4j}\left(e^{j2\pi 1010t} - e^{-j2\pi 990t} + e^{j2\pi 990t} - e^{-j2\pi 1010t}\right) \\
&= \frac{1}{2}\sin 2\pi\, 1010t + \frac{1}{2}\sin 2\pi\, 990t \\
&= \frac{1}{2}\cos\left(2\pi 1010t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(2\pi\, 990\, t - \frac{\pi}{2}\right)
\end{aligned}$$

Relationship between sinusoids and complex exponentials

$e^{j\theta} = \cos\theta + j\sin\theta$	$e^{-j\theta} = \cos\theta - j\sin\theta$
$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$	$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

A signal $x(t)$ is periodic with period T if $x(t - nT) = x(t)$ for any integer n.

A periodic signal can have multiple periods. The smallest positive period is the fundamental period. The fundamental frequency is f_0 is computed as $1/T_0$

[11:35-12:00] Fourier series

Any periodic signal $x(t)$ can be synthesized by adding complex exponentials with harmonic frequencies

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

where f_0 is the fundamental frequency.

Notice that

$$e^{j2\pi k f_0 t} = \cos(2\pi k f_0 t) + j\sin(2\pi k f_0 t)$$

Thus, the real component is periodic, the imaginary component is periodic, and both have the same periodicity.

A special case occurs with conjugate symmetric amplitudes $a_{-k} = a_k^*$, where $a_k = \frac{1}{2}A_k e^{j\phi_k}$, which leads to real-valued $x(t)$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k)$$

[12:00-12:50] Fourier series analysis and synthesis formulae

The Fourier series coefficients a_k can be computed from a periodic signal $x(t)$ by integration over one fundamental period T_0:

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_0 t}dt \qquad \text{(Fourier series analysis)}$$

A periodic signal can be synthesized by adding together complex exponentials weighted by the Fourier series coefficients a_k:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t} \qquad \text{(Fourier series synthesis)}$$

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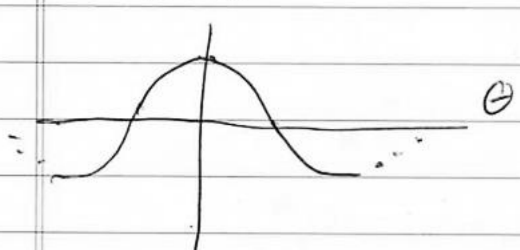
$$e^{j\theta} = \cos\theta + j\sin\theta$$

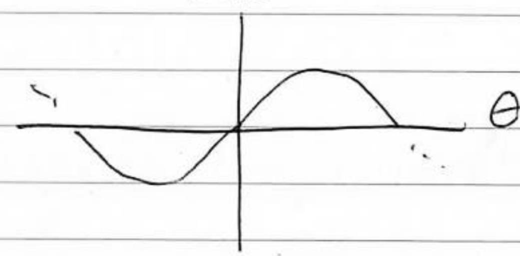
$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$$

$$= \cos\theta - j\sin\theta$$

$\cos\theta$



$\sin\theta$



Even symmetric about the origin / Odd (anti) symmetric about the origin

$$e^{j\theta} + e^{-j\theta} = (\cos\theta + j\sin\theta) + (\cos\theta - j\sin\theta)$$

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta \rightarrow \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{j\theta} - e^{-j\theta} = (\cos\theta + j\sin\theta) - (\cos\theta - j\sin\theta)$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta \rightarrow \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

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Synthesis $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

kth Harmonic is $f_k = k f_0$

$$x(t) = a_0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} a_k e^{j2\pi k f_0 t}$$

$$e^{j2\pi k f_0 t} = \cos(2\pi k f_0 t) + j\sin(2\pi k f_0 t)$$

periodic signal
- real component is periodic
- imaginary ~~component~~ component is periodic
- both have the same periodicity

Special case. Let $k=2$ & $k=-2$.

Select only two terms in ∞ sum $\Rightarrow$ $$a_{-2} e^{j2\pi(-2)f_0 t} + a_2 e^{j2\pi(2)f_0 t}$$

If $a_{-2} = a_2^*$ (conjugate symmetry)

$$a_2^* e^{j2\pi(-2)f_0 t} + \underbrace{a_2 e^{j2\pi(2)f_0 t}}_{\text{conjugate this term}}$$

gives this term $\leftarrow$

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Additional work after lecture.

$$a_2^* e^{j 2\pi(-2) f_0 t} + a_2 e^{j 2\pi (2) f_0 t}$$

Let $a_2 = r_2 e^{j\theta_2}$ in polar form

$$a_2^* = r_2 e^{-j\theta_2}$$

$$(r_2 e^{-j\theta_2}) e^{j 2\pi(-2) f_0 t} + (r_2 e^{j\theta_2}) e^{j 2\pi (2) f_0 t} =$$

$$r_2 e^{-j 2\pi (2) f_0 t - j\theta_2} + r_2 e^{j 2\pi (2) f_0 t + j\theta_2} =$$

$$r_2 e^{-j(2\pi(2) f_0 t + \theta_2)} + r_2 e^{j(2\pi(2) f_0 t + \theta_2)} =$$

$$r_2 \, 2 \cos(2\pi(2) f_0 t + \theta_2)$$

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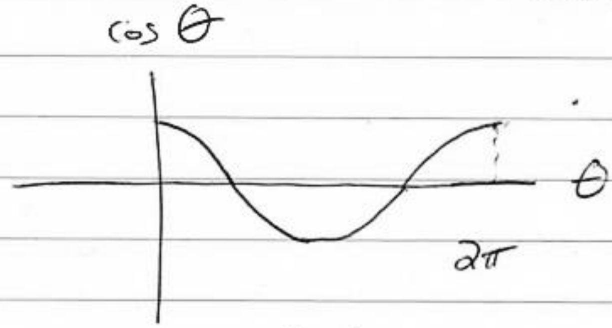
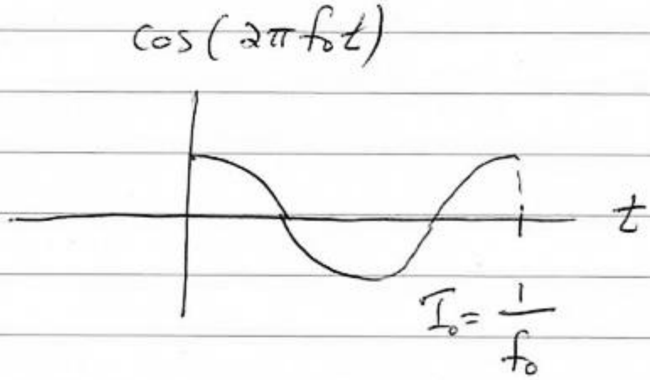




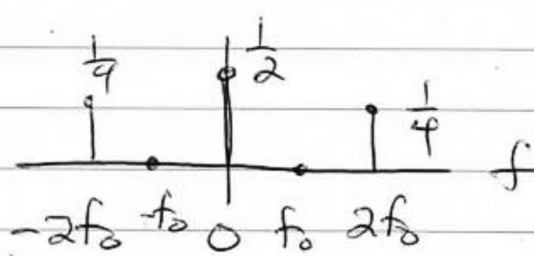
Average value is zero over one fundamental period.

$$\cos^2(2\pi f_0 t) = \frac{1}{2} + \underbrace{\frac{1}{2}\cos(2\pi(2f_0)t)}_{\text{average value is zero over the fundamental period}}$$

average value

As a Fourier series

$$\cos(2\pi(2f_0)t) = \frac{1}{2}e^{-j2\pi(2f_0)t} + \frac{1}{2}e^{j2\pi(2f_0)t}$$



Fourier Series Coefficients

$a_0 = \frac{1}{2}$

$a_2 = \frac{1}{4}$ $a_{-2} = \frac{1}{4}$